Some Properties of the Energy Load

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The combined results of previous theoretical investigations that used the Rayleigh-Ritz technique to determine the postbuckling behavior of thin shells seem to indicate that these approximate solutions are converging from above to the exact solution. The present report rigorously establishes this upper bound property for the particular case of the "energy load" of Friedrichs and Tsien. A new formulation of this load based on the derivation of a functional that has the energy load as its minimum leads to several other significant results: 1) it is shown that the energy load is positive, 2) an explicit lower bound to the energy load is derived, and 3) a sufficient condition for the existence of an energy load is established. These results are then demonstrated for the case of a cylindrical shell under axial compression.

Nomenclature

A = surface area of shell
$C^{\alpha\beta\gamma\delta}$ = stress-strain coefficients
D = $Eh^3/[12(1 - \nu^2)]$
E = Young's modulus
$E_{\alpha\beta}, N^{\alpha\beta}$ = components of strain and stress tensors
h = shell thickness
$K_{\alpha\beta}$ = components of curvature tensor
λ = load factor
L = shell length
ν = Poisson's ratio
R = shell radius
x, s = axial and circumferential coordinates
u, v, w = axial, circumferential, and normal displacements
U_b, U_m = bending and membrane energies

Introduction

THE large deflection analysis performed by von Kármán and Tsien[1] to determine the postbuckling characteristics of a cylindrical shell subjected to axial compression has been refined and extended by numerous authors.[2-4] In these investigations energy techniques were used to arrive at approximate postbuckling curves. The combined results, presented in Fig. 1, indicate that the inclusion of more degrees of freedom in the displacement function continuously lowers the large deflection equilibrium loads. Moreover, the lowest curve D calculated by Almroth[4] is in close agreement with the experimental data of Thielemann.[5] In view of this apparent convergence from above, it is not unreasonable to expect that energy techniques that yield upper bound approximations for the case of small deflections[6] may also provide upper bound solutions in the postbuckling range. A rigorous proof of the existence of this property, however, has not appeared in the literature.

The present investigation establishes this property for one point on the postbuckling curve. A sufficient condition for the existence of the upper bound property of the Rayleigh-Ritz solution is derived for the particular case of the "energy load" of Friedrichs[7] and Tsien.[8] This load was proposed as an explanation to the "jump" phenomenon observed in shell buckling. It is defined as the lower limit of all loads for which a buckled state of equilibrium exists with an energy level below that of the unbuckled state. In contrast to the von Kármán "lower buckling load," defined as the minimum equilibrium load in the buckled state, the "energy load" precludes the possibility of a shell jumping to a state of higher potential energy. The physical significance of this load was pointed out by Bodner and Gjelsvik[9] who found that for certain elastic systems the "energy load" is a lower bound to the snap buckling loads associated with a particular class of geometrical imperfections.

This investigation also presents a new formulation of the energy load which leads to several other significant results: 1) it is shown that the energy load is positive, 2) an explicit lower bound to the energy load is derived, and 3) a sufficient condition for the existence of an energy load is established. These general results are then demonstrated for the case of a cylindrical shell under axial compression.

Energy Load Functional

The potential energy of a shell (see Appendix) may be expressed in the following form:

$$V = A_2(\bar{w}) - \lambda W_2(\bar{w}) + 2B_3(\bar{w}) + C_4(\bar{w}) \tag{1}$$

in which the subscripts denote the order of $\bar{w}$ and its derivatives contained in the functionals A_2, W_2, B_3, and C_4, and λ is a load factor. For the normalized transverse deflection w defined by‡

$$\bar{w} = kw \qquad \text{with} \qquad W_2(w) = 1 \tag{2}$$

the potential energy becomes

$$V = k^2A_2(w) - \lambda k^2 + 2k^3B_3(w) + k^4C_4(w) \tag{3}$$

Equilibrium is obtained by requiring the vanishing of the first variation of the potential energy; this leads to the following two equations:

$$\partial V/\partial k = 0 \tag{4}$$

and

$$\delta_w V = 0 \qquad \text{subject to} \qquad W_2(w) = 1 \tag{5}$$

where $\delta_w V$ represents the first variation of V with respect to w. Carrying out these operations and using the notation presented in the Appendix, one obtains the two equations of

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‡ This normalization is always possible if the prebuckling states of plane stress exhibit no principal stresses in tension. If both tensile and compressive principal stresses occur, such as in the case of torsion, further precautions have to be introduced.

equilibrium in the form

$$A_2(w) - \lambda + 3kB_3(w) + 2k^2C_4(w) = 0 \qquad (6)$$

and

$$A_{11}(w, \beta) - \gamma W_{11}(w, \beta) + 2kB_{21}(w, \beta) + k^2C_{31}(w, \beta) = 0 \qquad (7)$$

for all β, subject to $W_2(w) = 1$, in which, for example, $\delta_w A_2(w) = A_{11}(w, \beta)$, and γ is a Lagrangian multiplier.

To describe the energy-load factor λ_T, the preceding equilibrium equations must be supplemented with the additional condition that the potential energies of the buckled and unbuckled states are equal. For the potential energy function V, defined in the Appendix as the difference in the potential energies of the buckled and unbuckled states, this is equivalent to setting $V = 0$ and leads to the following equation:

$$A_2(w) - \lambda + 2kB_3(w) + k^2C_4(w) = 0 \qquad (8)$$

The energy load is now described by the smallest load factor λ that permits a nontrivial solution of Eqs. (6–8). By solving Eqs. (6) and (8) simultaneously to find

$$k = -B_3(w)/C_4(w) \qquad (9)$$

and by substituting this value of k into Eqs. (7) and (8), one can reduce the number of relevant equations to the following two:§

$$A_{11}(w, \beta) - \gamma W_{11}(w, \beta) - (2B_3/C_4)B_{21}(w, \beta) + (B_3^2/C_4^2)C_{31}(w, \beta) = 0 \qquad (10)$$

for all β, and

$$\lambda = \Lambda \equiv A_2(w) - [B_3^2(w)/C_4(w)] \qquad (11)$$

subject to $W_2(w) = 1$.

Two related properties of the functional Λ in Eq. (11) will now be proved: 1) the energy-load factor λ_T is the lowest stationary value of this functional, and 2) this stationary value is the absolute minimum of Λ. The proof of the first proposition follows directly from the form of the stationary equation of the functional Eq. (11), namely,

$$\delta_w \Lambda = 0 \qquad (12)$$

which in its expanded form becomes

$$A_{11}(w, \beta) - \bar{\gamma} W_{11}(w, \beta) - (2B_3/C_4)B_{21}(w, \beta) + (B_3^2/C_4^2)C_{31}(w, \beta) = 0 \qquad (13)$$

for all β where $\bar{\gamma}$ is a Lagrangian multiplier. The eigenvalue $\bar{\gamma}$ is determined by setting $\beta = w$ in Eq. (13) to obtain

$$A_{11}(w, w) - \bar{\gamma} W_{11}(w, w) - (2B_3/C_4)B_{21}(w, w) + (B_3^2/C_4^2)C_{31}(w, w) = 0 \qquad (14)$$

With the aid of the identities given by Eqs. (A11) and (A12) of the Appendix, Eq. (14) can be solved for $\bar{\gamma}$ to yield

$$\bar{\gamma} = A_2(w) - [B_3^2(w)/C_4(w)] \equiv \Lambda \qquad (15)$$

by Eq. (11). Consequently, Eq. (13) becomes

$$A_{11}(w, \beta) - \Lambda W_{11}(w, \beta) - (2B_3/C_4)B_{21}(w, \beta) + (B_3^2/C_4^2)C_{31}(w, \beta) = 0 \qquad (16)$$

for all β. This is identical to Eq. (10) if γ is identified with Λ. Consequently, the stationary values of Λ and associated functions w satisfy the equilibrium equation, Eq. (10). The energy load factor λ_T is, therefore, the smallest stationary value of the functional Λ defined in Eq. (11), provided such a value exists.

§ Similar expressions have been derived[9] for the von Kármán "lower buckling load."

Existence of a Minimum

To show this existence and with it the existence of a minimum, one may rewrite the functional by using the definitions of A_2, B_3, and C_4 given in the Appendix by Eq. (A9). If at the same time one incorporates the notation of an inner product by

$$\left(\overset{i}{E}, \overset{j}{E}\right) = \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta}\,\overset{i}{E}_{\alpha\beta}\,\overset{j}{E}_{\gamma\delta}\,dA \qquad (i, j = 1, 2) \qquad (17)$$

in which the $\overset{i}{E}$ are defined in Eq. (A7), one arrives at the following revised form:

$$\Lambda = A_2 - (B_3^2/C_4) = U_b + \left(\overset{1}{E}, \overset{1}{E}\right) - \left[\left(\overset{1}{E}, \overset{2}{E}\right)^2 \Big/ \left(\overset{2}{E}, \overset{2}{E}\right)\right] \qquad (18)$$

Substitution of Schwarz's inequality

$$\left(\overset{1}{E}, \overset{2}{E}\right)^2 \leqslant \left(\overset{1}{E}, \overset{2}{E}\right)\left(\overset{2}{E}, \overset{2}{E}\right) \qquad (19)$$

into Eq. (18) yields, in view of $\left(\overset{2}{E}, \overset{2}{E}\right) > 0$,

$$\Lambda = A_2(w) - [B_3^2(w)/C_4(w)] \geq U_b(w) > 0 \qquad (20)$$

Therefore the minimum of U_b with respect to w, if it exists, is a lower bound to Λ; that is,

$$\Lambda(w) \geq U_{b_{\min}} > 0 \qquad (21)$$

for all w satisfying Eq. (2). On the other hand the functional Λ must have a minimum if it is bounded from below. Hence the existence of a minimum for U_b is a sufficient condition for the existence of an energy-load factor λ_T, and therefore the following inequalities hold:

$$\Lambda \geq \Lambda_{\min} = \lambda_T \geq U_{b_{\min}} > 0 \qquad (22)$$

for all w satisfying Eq. (2).

The physical significance of $U_{b_{\min}}$ is readily obtained by setting the first variation of $U_b(w)$, subject to the condition $W_2(w) = 1$, equal to zero. This leads to the equation

$$(h^2/12)C^{\alpha\beta\gamma\delta}w^{|}_{\alpha\beta\gamma\delta} - \lambda N^{\circ\alpha\beta}w^{|}_{\alpha\beta} = 0 \qquad (23)$$

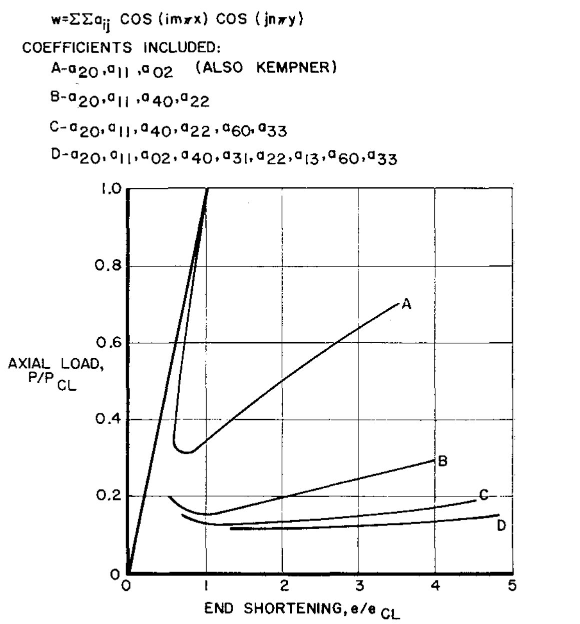

Fig. 1 Theoretical load-displacement curves for the axially-loaded cylinder.[4]

which is identical to the equilibrium equation for the buckling of a plate of width $2\pi R$ and length L having the same periodic boundary conditions prescribed on the shell. It therefore follows that

$$U_{b_{\min}} = \lambda_P \leq \lambda_T \tag{24}$$

where λ_P is the classical buckling load of the "equivalent" plate.

A Stable Configuration

The preceding results will now be used to establish the stability of the energy-load configuration. The equilibrium state at the energy load is stable if the following matrix is positive definite:

$$\begin{bmatrix} \delta_w{}^2V & \delta_w\delta_kV \\ \delta_w\delta_kV & \delta_k{}^2V \end{bmatrix} > 0 \tag{25}$$

Since

$$\delta_k{}^2V = (d^2V/dk^2) = (2B_3{}^2/C_4) > 0 \tag{26}$$

by Eqs. (3, 9, and 11) with ($C_4 > 0$), a necessary and sufficient condition for Eq. (25) to hold is

$$I_2 = \delta_w{}^2V\delta_k{}^2V - (\delta_w\delta_kV)^2 > 0 \tag{27}$$

At the energy load I_2 reduces to

$$I_2 = 2(B_3{}^4/C_4{}^3)\delta_w{}^2\Lambda \tag{28}$$

where $\delta_w{}^2\Lambda$ is the second variation of the energy-load functional Eq. (11). Since Λ takes on its minimum value at the energy load, it follows that

$$\delta_w{}^2\Lambda > 0 \tag{29}$$

and from Eq. (28) that $I_2 > 0$ which establishes that the energy-load configuration is a stable buckled state.

Rayleigh-Ritz Solution

The usual method for computing the energy load is to apply the Rayleigh-Ritz technique to the potential energy. Previous authors[7,9,11] have attempted to show that this approximation is an upper bound to the actual energy load. In all cases the arguments were based on the assumption that the energy-load configuration is a stable buckled state. A rigorous investigation to determine the conditions for which the energy-load configuration is indeed stable has not, however, appeared in the literature. In the present investigation a sufficient condition for stability has been established, i.e., the energy-load configuration is a stable state if the minimum to the bending energy exists. The importance of this condition is that it provides a rigorous means for establishing the existence of the upper-bound property of the Rayleigh-Ritz solution.

As an alternative approach, the Rayleigh-Ritz technique may also be applied to the energy-load functional Eq. (11). Since the energy load, if it exists, is the minimum of this functional, it is obvious that this approach will also lead to an upper bound. The advantage in working with Eq. (11), however, is the added simplicity that results from having to solve only one set of equations

$$\delta\Lambda = 0 \tag{30}$$

instead of the usual

$$\delta V = 0 \tag{31}$$

subject to

$$V = 0 \tag{32}$$

It is interesting to note that both formulations can be shown to yield the same approximation.

In the following section an approximate value of the energy load for a cylindrical shell under axial loading is obtained by applying the Rayleigh-Ritz technique to the energy-load functional. In addition it is shown that the approximate solution is an upper bound to the actual energy load.

Application to a Cylindrical Shell

The energy load for a cylindrical shell under axial loading will now be calculated for the zero-slope/zero-shear boundary conditions. For this case the minimum of the bending energy U_b in Eqs. (A3) exists and can be easily shown to equal[10]

$$U_{b_{\min}} = D\pi^2/hL^2P_E \tag{33}$$

in which P_E is the classical buckling load.¶ It therefore follows from Eqs. (22) and (33) that the energy load exists and is bounded from below:

$$P_T \equiv \lambda_TP_E \geq P_EU_{b_{\min}} = Eh^2\pi^2/[12(1-\nu^2)L^2] \tag{34}$$

This lower bound is believed to be considerably smaller than the actual energy load because of the presence of h^2/L^2. The dependence on L contradicts the observed buckling behavior which indicates an absence of length effects for sufficiently long shells. Moreover, the dependence on h^2 makes the lower-bound approximation of higher order in the thickness than previously calculated approximations.[8] Consequently, the deviation between the actual energy load and the lower bound in Eq. (34) becomes most pronounced for long thin shells.

The potential energy of an axially compressed cylindrical shell given by

$$V = \frac{D}{2}\int_A[(w_{,xx} + w_{,ss})^2 + 2(1-\nu)(w_{,xs}{}^2 - w_{,xx}w_{,ss})]dA - \frac{Ph}{2}\int_A w_{,x}{}^2dA + \frac{h}{4}\int_A\left[\phi_{,ss}w_{,x}{}^2 + \phi_{,xx}w_{,s}{}^2 - 2\phi_{,xs}w_{,x}w_{,s} - 2\phi_{,xx}\frac{w}{R}\right]dA \tag{35}$$

with the associated compatibility equation

$$\nabla^4\phi = E[w_{,xs}{}^2 - w_{,xx}w_{,ss} - (1/R)w_{,xx}] \tag{36}$$

will now be reduced to the form in the Appendix. In the usual manner first substitute the assumed radial deflection

$$w = \sum_{\alpha,\beta} a_{\alpha\beta}\cos\frac{\alpha m\pi x}{L}\cos\frac{\beta ns}{R} \tag{37}$$

which satisfies the zero-slope/zero-shear boundary conditions, and periodicity if m and n are integers into Eq. (36) to obtain the following solution for the stress function ϕ:

$$\phi = E\left[\sum_{\substack{\alpha\ \beta \\ \pm\gamma\pm\delta}} a_{\alpha\beta}a_{\gamma\delta}P_{\alpha\beta\gamma\delta}\cos\frac{(\alpha+\gamma)m\pi x}{L}\cos\frac{(\beta+\delta)ns}{R} + \sum_{\alpha\ \beta} a_{\alpha\beta}Q_{\alpha\beta}\cos\frac{\alpha m\pi x}{L}\cos\frac{\beta ns}{R}\right] \tag{38}$$

with

$$Q_{\alpha\beta} = \frac{\alpha^2\mu^2R}{n^2(\alpha^2\mu^2+\beta^2)^2}$$

$$P_{\alpha\beta\gamma\delta} = \frac{\alpha\delta(\gamma\beta - \alpha\delta)\mu^2}{4[(\alpha+\gamma)^2\mu^2 + (\beta+\gamma)^2]^2} \tag{39}$$

and

$$\mu = m\pi R/nL \tag{40}$$

¶ A minimum to the bending energy also exists for a simply-supported shell and has, in fact, the same value as that given in Eq.(33).

The notation

$$\sum_{\substack{\alpha \\ \pm\gamma}}$$

indicates that for α summed over the values 0 and 1, γ is summed over -1, -0, 0, and 1. In addition, the subscripts on $a_{\alpha\beta}$ are always considered nonnegative.

Substitution of Eqs. (37) and (38) into Eq. (35) brings the potential energy into the desired form, namely,

$$V = A_2(w) - \lambda P_E W_2(w) + 2B_3(w) + C_4(w) \tag{41}$$

where

$$A_2(w) = \frac{Dn^4\pi L}{4R^3} \sum_{\alpha\ \beta} a_{\alpha\beta}\left[(\alpha^2\mu^2 + \beta^2)^2 + \frac{E\alpha^2\mu^2 R}{Dn^2} Q_{\alpha\beta}\right] \tag{42}$$

$$W_2(w) = \frac{n^2\pi L}{4R} \sum_{\alpha\ \beta} a_{\alpha\beta}{}^2\alpha^2 \tag{43}$$

$$B_3(w) = \frac{E\mu^2 n^2 \pi L}{64R^3} \sum_{\substack{\alpha\ \beta \\ \pm\gamma\pm\delta \\ \pm\sigma\pm\phi}} a_{\alpha\beta}a_{\gamma\delta}a_{\sigma\phi}[n^2(\beta^2\gamma\sigma - 2\alpha\beta\gamma\phi + \alpha^2\delta\phi)Q_{\alpha\beta} + 8R(\alpha+\gamma)^2 P_{\alpha\beta\gamma\delta}] \tag{44}$$

subject to

$$\alpha \pm \gamma \pm \sigma = 0 \qquad \text{and} \qquad \beta \pm \delta \pm \phi = 0 \tag{45}$$

$$C_4(w) = \frac{E\mu^2 n^4 \pi L}{32R^3} \sum_{\substack{\alpha\ \beta \\ \pm\gamma\pm\delta \\ \pm\sigma\pm\phi \\ \pm\omega\pm\theta}} a_{\alpha\beta}a_{\gamma\delta}a_{\sigma\phi}a_{\omega\theta}[(\beta+\delta)^2\sigma\omega - 2\sigma\theta(\alpha+\gamma)(\beta+\delta) + (\alpha+\gamma)^2\phi\theta]P_{\alpha\beta\gamma\delta} \tag{46}$$

subject to

$$\alpha \pm \gamma \pm \sigma \pm \omega = 0 \qquad \text{and} \qquad \beta \pm \delta \pm \phi \pm \theta = 0 \tag{47}$$

The restrictions on the summation indices Eqs. (45) and (47) imply that the odd coefficients a_{rs}, i.e., $r + s$ equals an odd integer, must occur in pairs. As a consequence of this condition, it can be shown[10] that the equilibrium equations associated with the odd coefficients are satisfied if all the odd coefficients are identically zero. Therefore one needs to retain only the even coefficients in the series Eq. (37) as was done in the recent analysis of Almroth[4] and Cox.[12]

The energy-load functional Eq. (11) is now constructed using the series (42–47)

$$\Lambda(a_{\alpha\beta}, m, n) = [A_2 - (B_3{}^2/C_4)]/W_2 \tag{48}$$

and then minimized with respect to the nonzero (even) coefficients and the integer wave numbers m and n. This computation was performed on an IBM 7090 digital computer using the Newton-Raphson iteration scheme to solve the simultaneous nonlinear equations in $a_{\alpha\beta}$. These coefficients are then substituted into Eq. (48) to obtain the Rayleigh-Ritz approximation for the energy load $P_T{}^R$.

This technique was illustrated using nine even coefficients for the case $R = 4$, $L = 4$, and $h = 0.004$. Minimization of Eq. (48) yielded an approximate energy load equal to

$$P_T{}^R = 0.0983(Eh/R) \tag{49}$$

which is significantly lower than the approximate value

$$P_T{}^R = 0.238(Eh/R) \tag{50}$$

calculated by Tsien[8] using only three arbitrary coefficients. Moreover, by Eq. (22), the low value given by Eq. (49) is known to be an upper bound to the exact energy load.

For the preceding values of the parameters, the lower bound given in Eq. (34) equals

$$P_T \geq 0.0009(Eh/R) \tag{51}$$

which, as previously anticipated, is significantly lower than the upper bound approximation in Eq. (49).

Conclusions

This analysis has shown that the energy load is positive and exists if the minimum of the bending energy exists. A new formulation of the energy load also enabled a lower bound to the energy load to be explicitly derived. In addition, a sufficient condition is derived which rigorously establishes the upper-bound property of the Rayleigh-Ritz approximation for the energy load.

Appendix: Derivation of Basic Relations

The basic relations are derived by considering a general shell-buckling theory based on Donnell-type approximations.[13] The additional membrane-strain and curvature-displacement relationships will be taken in the form

$$\begin{aligned} E_{\alpha\beta} &= \tfrac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha - 2wb_{\alpha\beta} + w|_\alpha w|_\beta) \\ K_{\alpha\beta} &= w|_{\alpha\beta} \end{aligned} \tag{A1}$$

in which $\alpha = 1, 2$ refers to the middle surface coordinates, $b_{\alpha\beta}$ is the second fundamental form, and all differentiation is covariant. Then the additional potential energy, i.e., the difference in the potential energies of the buckled and unbuckled states, can be reduced to

$$V = U_b + U_m - \lambda W \tag{A2}$$

in which

$$\left.\begin{aligned} U_b &= \frac{h^3}{24}\int_A C^{\alpha\beta\gamma\delta} w|_{\alpha\beta} w|_{\gamma\delta} dA \geq 0 \\ U_m &= \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta} E_{\alpha\beta}E_{\gamma\delta} dA \geq 0 \\ W &= -\frac{h}{2}\int_A N^{\circ\alpha\beta} w|_\alpha w|_\beta dA \end{aligned}\right\} \tag{A3}$$

where the $N^{\circ\alpha\beta}$ are the prebuckling stresses and $C^{\alpha\beta\gamma\delta}$ are the usual stress-strain coefficients defined by

$$N^{\alpha\beta} = C^{\alpha\beta\gamma\delta}E_{\gamma\delta} \tag{A4}$$

with $N^{\alpha\beta}$ representing the additional membrane stresses.

The equilibrium equations for the middle surface stresses, derived by setting equal to zero the first variation with respect to u_α of the potential energy, are

$$N^{\alpha\beta}|_\alpha = 0 \tag{A5}$$

These equations are linear in u_α and linear and quadratic in w. Solving for u_α one formally obtains

$$u_\alpha(w) = \overset{1}{u}_\alpha(w) + \overset{2}{u}_\alpha(w) \tag{A6}$$

in which $\overset{1}{u}_\alpha$ and $\overset{2}{u}_\alpha$ are, respectively, first- and second-order functions in w and its derivatives. Substitution of this into the strain components Eq. (A1) yields

$$E_{\alpha\beta}(w) = \overset{1}{E}_{\alpha\beta}(w) + \overset{2}{E}_{\alpha\beta}(w) \tag{A7}$$

in which, according to the previous notation, $\overset{1}{E}_{\alpha\beta}$ and $\overset{2}{E}_{\alpha\beta}$ are first- and second-order functions in w.** Finally the substitution of Eq. (A7) into the potential energy gives a revised form that is a function of the transverse displacement only, namely,

$$V = A_2(w) - \lambda W_2(w) + 2B_3(w) + C_4(w) \tag{A8}$$

in which the subscript denotes the order of w and its derivatives contained in the functionals A_2, W_2, B_3, and C_4. These functionals are defined as follows:

$$\left.\begin{aligned} A_2(w) &= U_b(w) + \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta} \overset{1}{E}_{\alpha\beta} \overset{1}{E}_{\gamma\delta}\, da \geq 0 \\ W_2(w) &= W(w) \\ B_3(w) &= \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta} \overset{1}{E}_{\alpha\beta} \overset{2}{E}_{\gamma\delta}\, dA \\ C_4(w) &= \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta} \overset{2}{E}_{\alpha\beta} \overset{2}{E}_{\gamma\beta}\, dA \geq 0 \end{aligned}\right\} \tag{A9}$$

This notation, which is similar to that used by Koiter,[14] formally simplifies the analysis. Further simplification will result if the following related functionals are introduced:

$$A_2(a+b) = A_2(a) + A_{11}(a,b) + A_2(b)$$

and (A10)

$$B_3(a+b) = B_3(a) + B_{21}(a,b) + B_{12}(a,b) + B_3(b)$$

in which $B_{21}(a, b)$, for example, represents a functional that is second order in a and first order in b. Expansions for W_2 and C_4 can be similarly represented. Several useful identities can be derived from this expansion property, e.g.,

$$A_2(a) = A_2\left(\frac{a}{2} + \frac{a}{2}\right) = \tfrac{1}{4} A_2(a) + \tfrac{1}{4} A_{11}(a,a) + \tfrac{1}{4} A_2(a) \tag{A11}$$

or, after combining terms,

$$A_{11}(a,a) = 2A_2(a)$$

** Note that the former does not appear in plate theory.

In a similar manner the following relationships can be derived:

$$\left.\begin{aligned} W_{11}(a,a) &= 2W_2(a) \\ B_{21}(a,a) &= 3B_3(a) \\ C_{31}(a,a) &= 4C_4(a) \end{aligned}\right\} \tag{A12}$$

References

[1] von Kármán, T. and Tsien, H. S., "The buckling of thin cylindrical shells under axial compression," J. Aeronaut. Sci. **8**, 302–312 (1941).

[2] Kempner, J., "Post buckling behavior of axially compressed circular cylindrical shells," J. Aeronaut. Sci. **17**, 329–342 (1954).

[3] Leggett, D. M. A. and Jones, R. P. N., "The behavior of a cylindrical shell under axial compression when the buckling load has been exceeded," British Aeronautical Research Committee, Rand M2190 (1942).

[4] Almroth, B. O., "Postbuckling behavior of axially compressed circular cylinders," AIAA J. **1**, 630–633 (1963).

[5] Thielemann, W. G., "On the post-buckling behavior of thin cylindrical shells," NASA TND-1510, pp. 203–206 (1962).

[6] Temple, G. and Bickley, W. G., *Rayleigh's Principle and Its Applications to Engineering* (Dover Publications, Inc., New York, 1956), Chap. 1.

[7] Friedrichs, K. O., "On the minimum buckling load for spherical shells," *Theodore von Kármán Anniversary Volume* (California Institute of Technology, Pasadena, Calif., 1941), pp. 258–272.

[8] Tsien, H. S., "A theory for the buckling of thin shells," J. Aeronaut. Sci. **9**, 373–384 (1942).

[9] Bodner, S. R. and Gjelsvik, A., "Energy criterion and snap-buckling of arches," J. Eng. Mech. Div., Am. Soc. Civil Engrs. **88**, 87–134 (1962).

[10] Rubin, C. P., "A theoretical study of two buckling criteria for thin shells," Ph.D. Thesis, Univ. of Michigan, Ann Arbor, Mich. (1963).

[11] Langhaar, H. L. and Boresi, A. P., "Snap through and post buckling behavior of cylindrical shells under the action of external pressure," Univ. of Illinois Bull. Eng. Exp. Sta. Bull. 443 (1957).

[12] Cox, H. L., *The Buckling of Plates and Shells* (Pergamon Press, New York, 1963), Chap. 5.

[13] Koiter, W. T., "A consistent first approximation in the theory of thin elastic shells," *Proceedings of Symposium on the Theory of Thin Elastic Shells* (Interscience Publishers, Inc., New York, 1959).

[14] Koiter, W. T., "Elastic stability and post-buckling behavior," First All-Union Congress of Theoretical and Applied Mechanics, Moscow (1960).